

## SHORTER COMMUNICATIONS

### TRANSIENT SENSITIVITY COEFFICIENTS FOR THE THERMAL CONTACT CONDUCTANCE

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#### NOMENCLATURE

$c_p$ ,	specific heat;
$F$ ,	sum of squares function, equation (3);
$h$ ,	thermal contact conductance;
$k$ ,	thermal conductivity;
$L$ ,	thickness of specimen;
$t$ ,	time;
$T$ ,	calculated temperature;
$T_{\max}$ ,	maximum temperature;
$T_{\min}$ ,	minimum temperature;
$x$ ,	coordinate measured to right from left side of specimen 1;
$y$ ,	coordinate measured from interface pointing left;
$Y$ ,	measured temperature;
$z$ ,	coordinate measured from interface pointing right.

#### Greek symbols

$\rho$ ,	density;
$\phi$ ,	sensitivity coefficient;
$\phi$ ,	dimensionless sensitivity coefficients, equation (7).

#### INTRODUCTION

THE THERMAL contact conductance has been reported to vary with time in certain transient cases [1, 2]. In these papers the analyses were restricted either to thin specimens or to cases for which the temperatures at the interface were measured. By utilizing an effective method of analysis variously called nonlinear estimation, system identification, nonlinear least squares, etc., these restrictions can be removed. In analytically determining the most efficient designs for transient experiment, the sensitivity coefficients should be maximized. (The sensitivity coefficient is the derivative of the temperature with respect to the conductance.)

#### MATHEMATICAL DESCRIPTION

For the case of two specimens in thermal contact but with a finite conductance  $h$  at the interface, the heat-conduction problem can be mathematically described in part by

$$\frac{\partial}{\partial x} \left( k_n \frac{\partial T_n}{\partial x} \right) = \rho_n c_{p,n} \frac{\partial T_n}{\partial t} \quad (1)$$

$$\begin{aligned} -k_1 \frac{\partial T_1(L_1^-, t)}{\partial x} &= h[T_1(L_1^-, t) - T_2(L_1^+, t)] \\ &= -k_2 \frac{\partial T_2(L_1^+, t)}{\partial x}. \end{aligned} \quad (2)$$

In equation (1)  $n$  equals 1 or 2 depending on whether it applies to material 1, ( $0 < x < L_1$ ), or material 2, ( $L_1 < x < L_1 + L_2$ ). The interface is located at  $x = L_1$ . The effect of the finite value of  $h$  is not actually one which occurs in a plane; it is rather a volume effect and thus, in equations (1) and (2) the temperatures near the interface represent some sort of average over a plane normal to the  $x$ -coordinate.

The boundary conditions at  $x = 0$  and  $x = L_1 + L_2$  can be given  $T$ 's, given heat fluxes or other known conditions.

The method proposed herein involves minimizing with respect to  $h$  the summation

$$F(h) = \sum_i \sum_{j=M}^{M+J} (T_{ij} - Y_{ij})^2 \quad (3)$$

where  $T_{ij}$  and  $Y_{ij}$  are, respectively, the calculated and measured temperatures at  $(x_i, t_j)$ . The limits on  $j$  define a time interval for which a uniform  $h$  is to be found;  $h(t)$  can be considered to be approximated by constant-with-time segments or with linear segments, etc. The summation on  $i$  is over all the  $Y_{ij}$  except those serving as boundary or initial conditions.

The calculated temperatures  $T_{ij}$  can be obtained by a numerical solution of the problem which would permit (if desired)  $T$ -variable  $k$  and  $c_p$  (3, 4). A simple method for automatically minimizing  $F$  is given in (5); see also (6, 7). Briefly, one uses the Taylor series approximation

$$T_{ij}(h) \approx T_{ij}(h_i) + \phi_{ij} \Delta h_{i+1}, \quad (4)$$

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where

$$\Delta h_{l-1} = h_{l+1} + h_l \quad (5a)$$

$$\phi_{ij} = \frac{\partial T_{ij}(h_l)}{\partial h} \approx \frac{T_{ij}[h_l(1 + \epsilon)] - T_{ij}(h_l)}{\epsilon h_l} \quad (5b)$$

If  $\epsilon$  is made equal to 0.001 or smaller, the sensitivity coefficient  $\phi_{ij}$  is accurately approximated. The temperatures on the right-hand side of (5b) are calculated using a finite-difference program twice, with  $h_l$  and then with  $h_l(1 + \epsilon)$ . Using  $\partial F/\partial h = 0$  a correction in  $h$  is given by, (5),

$$\Delta h_{l+1} = \frac{\sum_i \sum_{j=M}^{M+J} [Y_{ij} - T_{ij}(h_l)] \phi_{ij}}{\sum_i \sum_{j=1}^{M+J} \phi_{ij}^2} \quad (6)$$

This iterative procedure begins with an estimated value of  $h_0$ , corresponding to  $l = 0$ , and continues for increasing values of the integer  $l$  until  $\Delta h_{l+1}/h_l$  is less than, say, 0.0005. If  $h$  is considered to be an unknown function of time and is approximated by constant- $h$  segments, the conductances are found corresponding to the time intervals between  $t_M$  and  $t_J$  with  $M$  and  $J$  taking on prescribed values.

#### INVESTIGATION OF SENSITIVITY COEFFICIENTS

It can be observed from (6) that the sensitivity coefficients  $\phi_{ij}$  figure prominently in the solution for  $h$ . From (4) note that the larger  $\phi_{ij}$  is, the more sensitive is  $T_{ij}$  to variations in  $h$ . This suggests, and is discussed in greater depth in (7), that  $h$  can be found most accurately when  $h\phi_{ij}/T_{ij}$  is maximized. A more convenient dimensionless quantity to maximize is

$$\bar{\phi} = \frac{h\phi}{T_{\max} - T_{\min}} \quad (7)$$

where  $T_{\max} - T_{\min}$  is the maximum temperature difference experienced by the specimens for the duration of the experiment.

To facilitate the investigation of  $\bar{\phi}$ , let material 2 be the same as material 1 and let  $L_1 = L_2 = L$ . A new coordinate system  $y$  starting at the interface and directed toward specimen 1 is introduced; let  $z$  be the coordinate starting at the interface and directed toward specimen 2. To further simplify the following discussion, assume that the thermal properties are temperature-independent and  $h$  is not a function of time or  $T$ .

The differential equation and interface relation for specimen 1 then are

$$k \frac{\partial^2 T_1}{\partial y^2} = \rho c_p \frac{\partial T_1}{\partial t} \quad (8)$$

$$k \frac{\partial T_1(0, t)}{\partial y} = h[T_1(0, t) - T_2(0, t)]. \quad (9)$$

The boundary condition at  $y = L$  can be either

$$T_1(L, t) = T_{1L}(t)$$

or

$$-k \frac{\partial T_1(L, t)}{\partial y} = q_1(t). \quad (10)$$

The initial condition is

$$T_1(y, 0) = T_{1i}(y). \quad (11)$$

The quantities  $T_{1L}$ ,  $q_1$  and  $T_{1i}$  are arbitrary, but known functions.

Taking the partial derivative of equations (8-11) with respect to  $h$  and then multiplying by  $h/(T_{\max} - T_{\min})$  gives

$$k \frac{\partial^2 \bar{\phi}_1}{\partial y^2} = \rho c_p \frac{\partial \bar{\phi}_1}{\partial t} \quad (12)$$

$$k \frac{\partial \bar{\phi}_1(0, t)}{\partial y} = h[\bar{\phi}_1(0, t) - \bar{\phi}_2(0, t)] + \frac{h[T_1(0, t) - T_2(0, t)]}{T_{\max} - T_{\min}} \quad (13)$$

$$\bar{\phi}_1(L, t) = 0 \quad \text{or} \quad \frac{\partial \bar{\phi}_1(L, t)}{\partial y} = 0 \quad (14)$$

$$\bar{\phi}_1(y, 0) = 0. \quad (15)$$

These equations apply for specimen 1, but a similar set for specimen 2 is obtained by interchanging 1 and 2 and by replacing  $y$  with  $z$ .

Notice that each equation from (12) through (15) is homogeneous except (13). Evidently if the temperature difference in (13) were zero, the sensitivity coefficient  $\phi_1$  would be equal to zero for all times. If  $T_1(0, t) - T_2(0, t)$  is positive, then this term in (13) is similar in effect to a plane heat sink in specimen 1 but as a heat source in specimen 2. In other words, if there is a heat flow from specimen 1 to specimen 2,  $\bar{\phi}$  is negative in specimen 1 and positive in specimen 2. According to equation (4) this results in the calculated temperatures in specimen 1 being decreased when  $h$  is increased.

This analogy with heat sources provides a basis for some insight into the behavior of the derivative  $\phi$  and hence, for the design of optimum experiments. For example, one learns that (a)  $|\phi|$  is greatest near the interface and decreases with distance from the interface, and (b) the larger  $(T_1 - T_2)/(T_{\max} - T_{\min})$  is across the interface, the larger  $|\phi|$  will be. From (a) one can deduce that thermocouples yield more information relative to  $h$  when they are located "near" the interface rather than "far" from it. In other words, a thermocouple positioned "near" the interface would usually yield more accurate values of the conductance than a thermocouple located at, say,  $y = L$ .

While (a) indicates, to some extent, where thermocouples

should be located, (b) suggests that certain experiments should be more effective than others. It suggests specifically regarding (i) the initial temperature distribution and (ii) the boundary conditions. Two initial distributions could be considered: uniform and equal temperatures in both specimens and uniform but unequal  $T$ 's in the specimens. For the case of uniform, equal temperatures, the specimens must be heated at either  $y = L$  or  $z = L$ . Consider, for example, the case for which the  $T$  at  $y = L$  takes a step change to  $T_{\max}$  and the  $T$  at  $z = L$  is maintained at  $T_{\min}$ . Compare this case with another with the same boundary conditions but with the initial temperature in specimen 1 equal to  $T_{\max}$  and the initial  $T$  in specimen 2 equal to  $T_{\min}$ . From a knowledge of heat-conduction phenomena one can deduce that the temperature-difference across the interface of the latter case will always be equal to or greater than for the former; hence, the latter is more effective for determining the thermal conductance. If any other experiment were imagined with the same boundary conditions at  $y = L$  and  $z = L$ , but with an arbitrary initial temperature distribution satisfying the condition that all the temperatures fall between  $T_{\min}$  and  $T_{\max}$ , one would still conclude that at any given time and position  $|\bar{\phi}|$  would be larger for the experiment mentioned above.

One should not conclude since the interface temperature drop should be maximized that this implies  $h$  itself should be made small. Since  $h$  is a "property", one cannot arbitrarily make  $h$  small. Furthermore, whether  $h$  is large or small (i.e.  $hL/k \gg 1$  or  $hL/k \ll 1$ ), the same conclusions apply although, if possible, (as will be shown in a forthcoming paper) it is best to make  $L < k/2h$ . In the forthcoming paper it will also be demonstrated for realistic cases that the percentage error in  $h$  is reduced when the criterion  $|\bar{\phi}|$  is maximized.

Consider now the effect of the temperature boundary conditions upon  $\phi$ . From the boundary conditions on  $\phi$  given by (14), note that a prescribed temperature history at  $y = L$  results in  $\phi$  being equal to zero at  $y = L$ ; in contrast note that a prescribed heat flux or insulation condition at  $y = L$  allows  $\phi$  to be non-zero. In general then, values of  $\phi$  near  $y = L$  will be greater for a flux condition than a prescribed temperature. It is not so obvious, but true, that for a substantial time  $\phi$  tends to be larger for the  $q$ -condition than the  $T$ -condition even near the interface. Hence, the desired boundary conditions at  $y = L$  and  $z = L$  are given heat fluxes or the simpler insulation condition. (Since a digital computer is to be used, heat transferred to the insulating materials at  $y = L$  and  $z = L$  can be included in the analysis by treating the specimens as composites.)

By comparing equations (12) through (15) with a comparable set of equations for specimen 2 one finds that

$$\bar{\phi}_1(y, t) = -\bar{\phi}_2(z, t) \quad (16)$$

when  $z = y$  and for the same  $t$ . This applies only when

specimens 1 and 2 are the same material with  $T$ -independent properties, have the same thickness, and are subject to the same type of boundary condition at both  $y = L$  and  $z = L$ —that is,  $T$  or  $q$ , but not both. Even so, equation (16) applies for a surprisingly wide class of problems. For example, the initial temperature distributions are arbitrary, as are the time variations of the boundary conditions.

The case of the time-varying conductance can be investigated by extending some of the above ideas. Note only the effect of small time intervals for determining a single  $h$ . Equations (12–15) are valid for finding the first  $h$ . Because of the diffusion nature of these equations, there is a finite time interval  $t_D$  before any thermocouple located inside the specimens is significantly affected by the interface. If the time interval used for determining  $h$  is smaller than  $t_D$ , then  $h(t)$  would be inaccurate and possibly even numerically unstable.

In conclusion, a method is given for determining the thermal contact conductance as a function of time and some points are made relative to an efficient design from an examination of the sensitivity coefficients.

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